*International Journal of Theoretical Physics, Vol. 43, No. 4, April 2004 (© 2004)*

# Dirac Equation and Its Solution Around Schwarzschild–de Sitter Black Hole

**Yan Lyu[1,3] and Yuan-Xing Gui[2]**

This paper aims to solve the radial parts of Dirac equation between the inner and the outer horizon in Schwarzschild–de Sitter geometry numerically. A "logarithm" approximation which almost has the same nature with the modified "tortoise" coordinate $r_*$ even close to the two horizons is found. It is used to replace the modified "tortoise" coordinate $r_*$, this leads to a simple analytically invertible relation between $r_*$ and the radius $r$. Then, the potential $V(r_*)$ is replaced by a collection of step functions. By a quantum mechanical method, the solution of the wave equation as well as the reflection and transmission coefficients are computed. The resulting wave turns out to be not close to that of a harmonic wave globally.

**KEY WORDS:** Dirac equation; Schwarzschild-de Sitter spacetime.

## 1. INTRODUCTION

Since Chandrasekhar separated the Dirac equation in Kerr geometry into radial and angular parts (Chandrasekhar, 1976, 1983), many authors have studied the curved spacetime outside a black hole. For example, the behavior of Dirac particles around Schwarzschild and Kerr black hole has been studied by Chakrabarti *et al.* who calculated the reflection and transmission coefficients as well as the solutions of incoming Dirac wave numerically (Chakrabarti and Mukhopadhyay, 2000a,b; Mukhopadhyay and Chakrabarti, 1999, 2000). While the black hole with cosmological horizon is much less studied. One example is the study of Brevik and Simonsen about the scalar field equation in Schwarzschild–de Sitter(SdS for short) spacetime. They approximated tortoise coordinate $x(r)$ by a tangent function $\tilde{x}(r)$ and replaced the potential function $V(r)$ by $V(x)$ after indentifying $\tilde{x}(r)$ and $x(r)$. Then, the scalar field equation is solved and the solution is close to that of a harmonic wave (Brevik and Simonsen, 2001).

[1] College of Physics and Technology, Shenyang Normal University, Liaoning Shenyang 110034, People's Republic of China.
[2] Department of Physics, Dalian University of Technology, Liaoning Dalian 116024, People's Republic of China.
[3] To whom correspondence should be addressed; e-mail: yanlvthp@yahoo.com.cn.

In the present paper, the basic Dirac equation in a static spherically symmetric spacetime is presented and as an example we solve the equation in SdS spacetime. A new method that a simple "logarithm" approximation is used to invert the relation between the radius $r$ and the modified tortiose coordinate $r_*$ so as to express the potential as a function of $r_*$ is adopted. This approximation is much valid because the logarithm function almost has the same nature with the modified tortoise coordinate globally. The potential is replaced by a collection of step functions in succession. Then, a quantum mechanical method (Chakrabarti and Mukhopadhyay, 2000a,b) is used to calculate the reflection and transmission coefficients and the wave functions numerically. Two cases are considered here also, one is when the two horizons are wide separated, the other is when the two horizons are lying close to each other. Then, the results of the two cases will be compared.

We adopt the signature $(- + ++)$ and put $\hbar = c = G = 1$.

## 2. THE BASIC EQUATION IN A STATIC SPHERICALLY SYMMETRIC SPACETIME

The line element for a static spherically symmetric spacetime is written as

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 \tag{1}$$

Following Chandrasekhar (1976, 1983), we get the Dirac equation in this spacetime and separate it into radial and angular equations. The coupled radial equations are given by

$$\Delta^{\frac{1}{2}} D_0 R_{-\frac{1}{2}} = (\lambda + imr)\Delta^{\frac{1}{2}} R_{-\frac{1}{2}} \tag{2a}$$

$$\Delta^{\frac{1}{2}} D_0^{\dagger} \Delta^{\frac{1}{2}} R_{+\frac{1}{2}} = (\lambda - imr) R_{-\frac{1}{2}} \tag{2b}$$

and the angular equations can be written as

$$L_{\frac{1}{2}} S_{+\frac{1}{2}} = -\lambda S_{-\frac{1}{2}} \tag{2c}$$

$$L_{\frac{1}{2}}^{\dagger} S_{-\frac{1}{2}} = \lambda S_{+\frac{1}{2}} \tag{2d}$$

where

$$D_n = \partial_r + \frac{ir^2\sigma}{\Delta} + n\frac{1}{\Delta}\frac{d\Delta}{dr} \tag{3a}$$

$$D_n^{\dagger} = \partial_r - \frac{ir^2\sigma}{\Delta} + n\frac{1}{\Delta}\frac{d\Delta}{dr} \tag{3b}$$

$$L_n = \partial_\theta + n\cot\theta + m\,\mathrm{cosec}\,\theta \tag{3c}$$

$$L_n^\dagger = \partial_\theta + n \cot\theta + m \operatorname{cosec}\theta \tag{3d}$$

$$\Delta \equiv r^2 f(r) \tag{3e}$$

Here $n$ is an integer, $\sigma$ is the frequency of the incoming Dirac wave, $m$ is the rest mass of the Dirac particle, and $R_{\pm\frac{1}{2}}$ are the radial wave functions corresponding to spin $\pm\frac{1}{2}$. $S_{\pm\frac{1}{2}}$ are the angular functions. $\Delta$ is called horizon function. From Eq. 2(a)–(d) and Eq. 3(a)–(d) we can see that the horizon function only affects the radial parts of the separated equations.

The coupled radial equations can be reduced to one-dimensional wave equations as

$$\left(\frac{d^2}{dr_*^2} + \sigma^2\right) Z_\pm = V_\pm Z_\pm \tag{4}$$

where

$$\begin{aligned} V_\pm = {} & \frac{\Delta^{\frac{1}{2}}(\lambda^2 + m^2 r^2)^{\frac{3}{2}}}{[r^2(\lambda^2 + m^2 r^2) + \lambda m \Delta/2\sigma]^2} [\Delta^{\frac{1}{2}}(\lambda^2 + m^2 r^2)^{\frac{3}{2}} \pm (P(\lambda^2 + m^2 r^2) \\ & + 3m^2 r \Delta)] \mp \frac{\Delta^{\frac{3}{2}}(\lambda^2 + m^2 r^2)^{\frac{5}{2}}}{[r^2(\lambda^2 + m^2 r^2) + \lambda m \Delta/2\sigma]^3} [2r(\lambda^2 + m^2 r^2) \\ & + 2m^2 r^3 + \lambda m P/\sigma] \end{aligned} \tag{5}$$

where $P = \frac{1}{2}\frac{d\Delta}{dr}$.

Tortoise coordinate is defined as

$$r' = \int \frac{dr}{f(r)} \tag{6}$$

and the modified tortoise coordinate is

$$r_* = r' + \frac{1}{2\sigma} \tan^{-1}(mr) \tag{7}$$

Eq. (4) is nothing but one-dimensional Schrödinger equation corresponding to the total energy of the wave $\sigma^2$ and potential energy $V_\pm$. It is clear that we obtain well-behaved functions by using $r'$ (and $r_*$) instead of $r$. The horizon(s) corresponding to $f(r) = 0$ is (are) shifted to infinity. On the other hand, Eq. (4) is a nonlinear equation because of the complicated potential function, we cannot give its analytical solution in any spacetime exactly. Only in the limiting case when $V \to 0$, the analytical solution can be given approximately (Khanal, 1984; Khanal and Panchapakesan, 1980). Here, we will use a numerical method to solve the wave functions in SdS spacetime in the whole range of $r_*$.

## 3. SOLUTION OF THE WAVE EQUATION IN SdS SPACETIME

In SdS spacetime $f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2$, where $M$ is the mass of the black hole. According to the above results, the cosmological constant $\Lambda$ which is included in the horizon functon only affects the radial parts of Dirac equation (Eq. 2(a) and 2(b)). The angular equations (Eq. 2(c) and 2(d)) have the same form with that in Schwarzschild geometry. The eigenvalue of the angular equation for spin $\pm\frac{1}{2}$ is obtained as $\lambda = (\ell + \frac{1}{2})^2$ (Chakrabarti, 1984; Goldberg *et al.*, 1967; Newman and Penrose, 1966), where $\ell$ is the orbital quantum number and $\lambda$ is the separation constant. Here, we choose $\ell = \frac{1}{2}$ and then $\lambda = 1$.

The tortoise coordinate in SdS spacetime is

$$r' = \frac{1}{2\kappa_e}\ln\left(\frac{r}{r_e} - 1\right) - \frac{1}{2\kappa_c}\ln\left(1 - \frac{r}{r_c}\right) + \frac{1}{2\kappa_o}\ln\left(1 - \frac{r}{r_o}\right) \tag{8}$$

and the modified tortoise coordinate is

$$r_* = r' + \frac{1}{2\sigma}\tan^{-1}(mr) \tag{9}$$

where $r_e$ is the black hole radius, $r_c$ is the cosmological radius, $r_o = -(r_e + r_c)$ which appears to be of no physical significance, and

$$\kappa_e = \frac{(r_c - r_e)(r_e - r_o)}{6r_e}\Lambda \tag{10a}$$

$$\kappa_c = \frac{(r_c - r_e)(r_c - r_o)}{6r_c}\Lambda \tag{10b}$$

$$\kappa_o = \frac{(r_o - r_e)(r_c - r_o)}{6r_o}\Lambda \tag{10c}$$

If the system approaches that of a Schwarzschild black hole, $r_e \to 2M$, $r_o \to -r_c = -\sqrt{\frac{3}{\Lambda}}$, so that $\kappa_e = \frac{1}{4M}$, $\kappa_c = \kappa_o = \sqrt{\frac{\Lambda}{3}}$ (Brevik and Simonsen, 2001), then the tortoise coordinate is reduced to

$$r' = r + 2M\ln(r - 2M) \tag{11}$$

To solve the wave equation (4), we need $V = V(r_*)$. As mentioned earlier, Eq. (9) for $r_*(r)$ thus must be inverted. In this context, a "logarithm" approximation will be available. The parameters are chosen as $M = m = \sigma = 1$ in the following calculations.

### 3.1. The Horizons Are Widely Separated, $\Lambda = 0.001$

The value of the cosmological constant must be small when the two horizons are far apart. We choose $\Lambda = 0.001$ as an example corresponding to $r_e = 2.0027$, $r_c = 53.7435$. The modified tortoise coordinate $r_*$ versus $r$ is shown in Fig. 1. It is

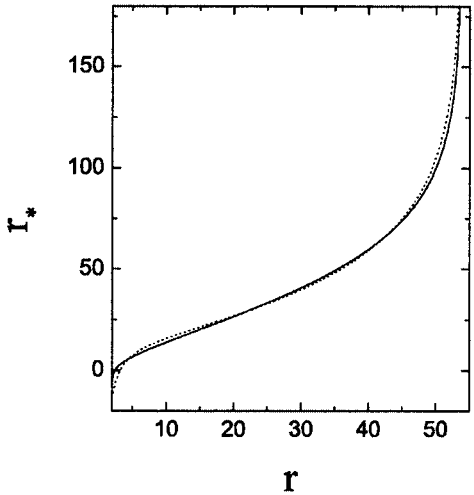


**Fig. 1.** Modified tortoise coordinate $r_*$ versus $r$ (solid line), together with the logarithm approximation $\tilde{r}_*$ (dotted line). $\Lambda = 0.001$.

approximated by a logarithm function in the bulk region between the two horizons. The relation

$$\tilde{r}_*(r) = a \ln \frac{r - r_e + b}{(r_c - r)r} + c \tag{12}$$

in which $a = 35$, $b = 0.9$, and $c = 152$ is a simple and valid approximation. Only close to the inner horizon the approximation Eq. (12) fails because the modified tortoise coordinate falls abruptly on the inner horizon. The approximation (Eq. (12)) will have the same infinity property with $r_*$ on the inner horizon if constant $b$ is chosen as zero, but $\tilde{r}_*$ is not wanted to deviate from $r_*$ too much in the whole range of $r$, then constant $b$ is chosen as 0.9. The inversion of Eq. (12) is easily done analytically as $r(\tilde{r}_*) = \frac{1}{2}\{r_c - \exp(\frac{c-\tilde{r}_*}{a}) + \sqrt{[r_c - \exp(\frac{c-\tilde{r}_*}{a})]^2 - 4\exp(\frac{c-\tilde{r}_*}{a})(b - r_e)}\}$. We substitute it into Eq. (5) and identify $\tilde{r}_*$ and $r_*$, thus obtain an approximation expression for $V_+ = V_+(r_*)$. The variation of $V_+(r_*)$ versus $r_*$ is shown in Fig. 2, it has been replaced by a collection of step functions. In fact, we use as many as 10,000 steps to accurately follow the shape of the potential in the following calculations so that the steps become indistinguishable from the actual potential function. The wave function on each step is given by quantum mechanics as

$$Z_{+,n} = A_n \exp[ik_n r_{*,n}] + B_n \exp[-ik_n r_{*,n}] \tag{13}$$

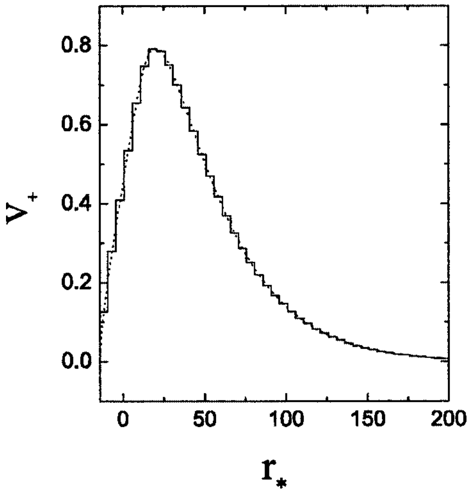


**Fig. 2.** Behavior of $V_+(r_*)$ (smooth dotted line). It is approximated as a collection of steps. $\Lambda = 0.001$.

Here $k$ is the wave number ($k = \sqrt{\sigma^2 - V_+}$) and $k_n$ is its value at $n$th step. The following standard junction conditions are included to ensure the function smooth

$$Z_{+,n} = Z_{+,n-1} \tag{14}$$

$$\left.\frac{dZ_+}{dr_*}\right|_n = \left.\frac{dZ_+}{dr_*}\right|_{n-1} \tag{15}$$

The "local reflection and transmission coefficients" on each step have the forms

$$R_n = \left|\frac{B_n}{A_n}\right|^2$$
$$= \left|\frac{\left(1 - \frac{k_{n-1}}{k_n}\right) A_{n-1}\exp[ik_{n-1}r_{*,n-1}] + \left(1 + \frac{k_{n-1}}{k_n}\right) B_{n-1}\exp[-ik_{n-1}r_{*,n-1}]}{\left(1 - \frac{k_{n-1}}{k_n}\right) A_{n-1}\exp[ik_{n-1}r_{*,n-1}] + \left(1 - \frac{k_{n-1}}{k_n}\right) B_{n-1}\exp[-ik_{n-1}r_{*,n-1}]}\right|^2 \tag{16}$$

$$T_n = 1 - R_n \tag{17}$$

The convergent value of $R_n$ is the reflection coefficient, i.e., $R = R_n|_{r_*\to+\infty}$. The transmission coefficient is: $T = 1 - R$. Here the no-reflection inner boundary condition is used: $R_n \to 0$ at $r_* \to -\infty$. The reflection and transmission coefficients turn out to be

$$R = 8.76 \times 10^{-5}, \quad T = 0.9999124 \tag{18}$$

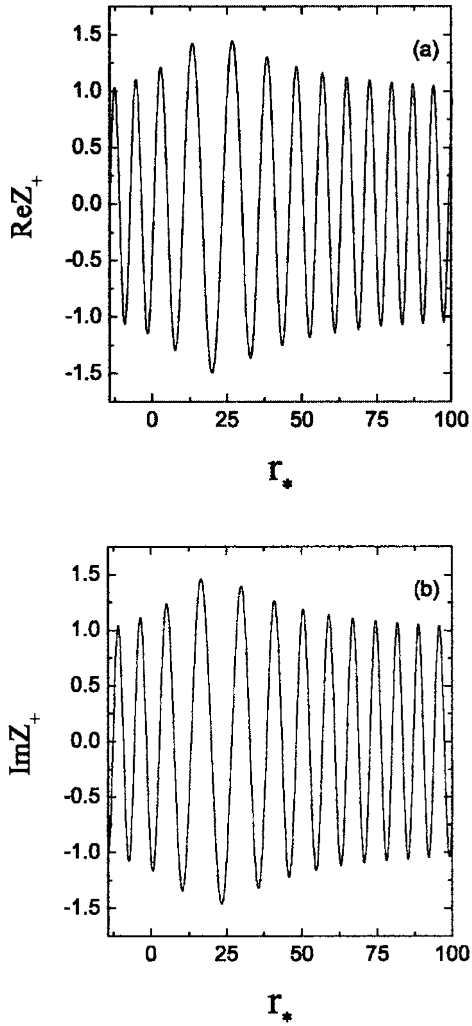


**Fig. 3.** (a) Amplitude of Re($Z_+$) of wave as a function of $r_*$. (b) Amplitude of Im($Z_+$). $\Lambda = 0.001$.

The solution (amplitude of Re($Z_+$) and Im($Z_+$)) is shown in Fig. 3(a) and 3(b). It is obvious that the amplitude as well as the wavelength of the wave functions remain constants close to the two horizons where $V_+ \to 0$. While there are apparent variations of the amplitude and the wavelength in the whole range. The amplitude has a maximum as $r_* \to 20$ because of the high potential in this region.

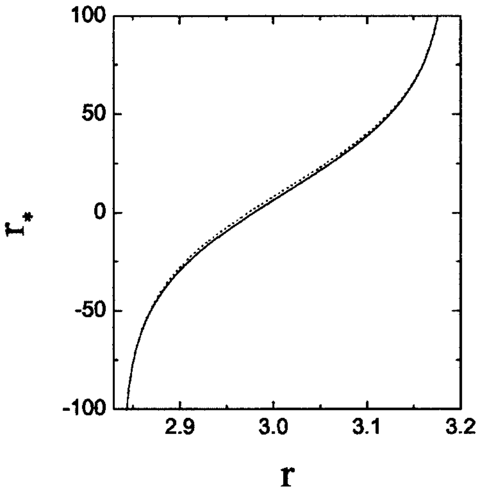


**Fig. 4.** Modified tortoise coordinate $r_*$ versus $r$ (solid line), together with the logarithm approxmation $r_*$ (dotted line). $\Lambda = 0.11$.

### 3.2. The Second Case, $\Lambda = 0.11$

In this case, the two horizons are close to each other. Correspondingly, $r_e = 2.8391$ and $r_c = 3.1878$. Again, a logarithm approximation to the potential is available. Figure 4 shows the modified tortoise coordinate $r_*(r)$ (solid line) and the approximation (dotted line) calculated from

$$\tilde{r}_*(r) = a' \ln \frac{r - r_e + b'}{(r_c - r)} + c' \tag{19}$$

where $a' = 26.5$, $b' = 0$, and $c' = 40$. The logarithm fnction presents a good approximation in the whole range even close to the horizons. In fact, the logarithm curve (dotted) is drawn shifting vertically by 1 unit for clarity because the two lines are so agreed that we cannot distinguish them. The potential $V_+ = V_+(r_*)$ is shown in Fig. 5. The potential barrier is much lower and wider than that in the first case. We also replace the potential barrier by a series of step functions. The same quantum mechanical method and parameters are used here, then the reflection and transmission cofficients are obtaned as

$$R = 2.9 \times 10^{-11}, \quad T \approx 1 \tag{20}$$

As expected, the reflection coefficient is lower than that in the first case, since the potential is lower. Figure 6(a) and 6(b) illustrates the behavior of the field amplitudes in an arbitraty chosen region between the horizons. The solution is close to that of a harmonic wave. While the wavelength and the amplitude should

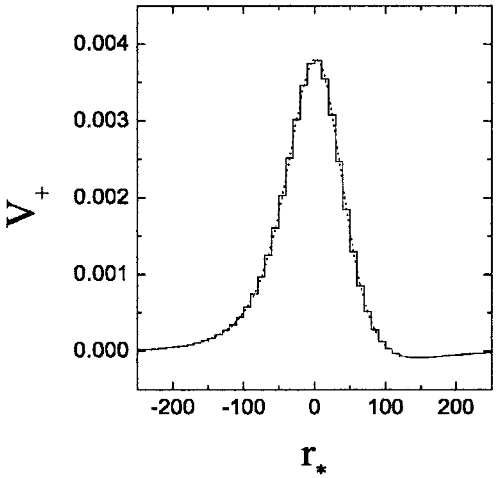


**Fig. 5.** Behavior of $V_{+}(r_{*})$ (smooth dotted line). It is approximated as a collection of steps. $\Lambda = 0.11$.

have a maximum near $r_{*} = 0$ where the potential has a maximum value, but this phenomenon is not obvious because of the low potential in the whole range.

## 4. CONCLUSION AND DISCUSSION

In this paper, we studied the scattering of Dirac paticles from a SdS black hole, particularly the nature of the radial wave functions and the reflection and transmission coefficients. The complicated SdS potential $V(r)$ in Eq. (5) can be written as a function of $r_{*}$ by means of simple, invertible approximate relationships between $r_{*}$ and $r$ (i.e., Eq. (12)) in the case when $\Lambda = 0.001$, and Eq. (19) in the case when $\Lambda = 0.11$). From Figs. 1 and 4 one sees that the logarithm approximation coincides with the original line (Eq. (9)) well. Then, a quantum mechanical step potential approach is used to calculate the "local reflection and transmission coefficients" and the wave fnctions. The reflection coefficient which turns out to be the convergent value of the local reflection coefficient is quite small especially in the second case ($\Lambda = 0.11$) because of the low potential barrier. Thus, practically all waves emanating from the outer horizon can propagate through the barrier to reach the inner horizon. The wavelength and the amplitude of the wave turn out to be larger near the region where the potential has a maximum value than anywhere else when $\Lambda = 0.001$. The wave should have the similar nature for the case $\Lambda = 0.11$, while the phenomenon is not obvious in this case because of the much less pronounced barrier-like property. Thus, the value of cosmological constant can seriously affect the nature of the wave and the reflection and transmission

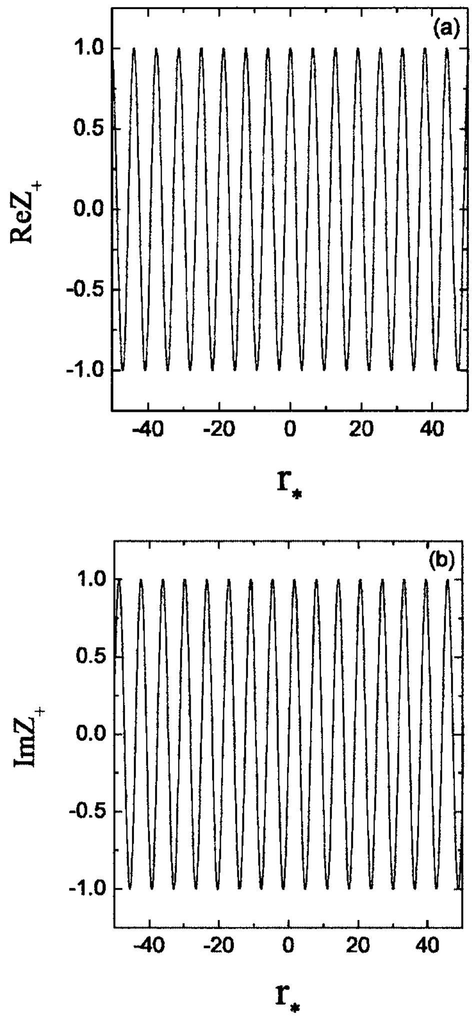


**Fig. 6.** (a) Amplitude of Re($Z_+$) of wave as a function of $r_*$. (b) Amplitude of Im($Z_+$). $\Lambda = 0.11$.

coefficients. There is something to be noted that the solutions corresponding to potential $V_-(r_*)$ can also be obtained in the same way and is not given in this paper.

We have chosen $M = m = \sigma = 1$ in order to calculate the solution easily. Then, the value of the potential $V(r_*)$ is much lower than the energy of the incoming particle $\sigma^2$. That is, we are considering the Dirac field equation in the high-frequency regime. The smallness of the calculated reflection coefficients is related to this fact. On the other hand, it will be desirable to study the low-frequency case also. While Eq. (9) should be approximated by a new logarithm function when the parameters are chosen differently because the paremeters $m$ and $\sigma$ are included in Eq. (9). The logarithm approximation used here is much efficient because it coincides with the modified tortoise coordinate very well even close to the two horizons, i.e., $\tilde{r}_* \to -\infty(+\infty)$ as $r \to r_e(r_c)$ especially in the case $\Lambda = 0.11$. The results in this paper can be used to discuss the Hawking radiation in further.

## REFERENCES

Brevik, I., and Simonsen, B. (2001). *General Relativity and Gravitation* **33**, 1839.
Chakrabarti, S. K. (1984). *Proceedings of the Royal Society of London A* **391**, 27.
Chakrabarti, S. K., and Mukhopadhyay, B. (2000a). *Monthly Notices of the Royal Society of London* **317**, 979.
Chakrabarti, S. K., and Mukhopadyay, B. (2000b). *Nuovo Cimento B* **115**, 885.
Chandrasekhar, S. (1976). *Proceedings of the Royal Society of London A* **349**, 571.
Chandrasekhar, S. (1983). *The Mathematical Theory of Black Holes*, Oxford University Press, New York.
Goldberg, J. N. *et al*. (1967). *Journal of Mathematical Physics* **8**, 2155.
Khanal, U. (1984). *Physical Review D* **32**, 870.
Khanal, U., and Panchapakesan, N. (1980). *Physical Review D* **24**, 829.
Mukhopadhyay, B., and Chakrabarti, S. K. (1999). *Classical and Quantum Gravity* **16**, 3165.
Mukhopadhyay, B., and Chakrabarti, S. K. (2000). *Nuclear Physics B* **582**, 627.
Newman, E., and Penrose, R. (1966). *Journal of Mathematical Physics* **7**, 863.